“To explain all nature is too difficult a task for any one man or even for any one age. ‘Tis much better to do a little with certainty and leave the rest for others that come after you.”

~ Isaac Newton
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Mission Statement

The Haverford School’s math journal is designed to enhance the interests, talents, and achievements of all individuals in the mathematical sciences and promote the work of those most passionate about the discipline. The following articles were written by members of the Haverford community and edited by math journal staff. We hope these articles inspire readers to further explore the universally beautiful realm of mathematics.

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Editors-in-Chief: Eusha Hasan, Mickey Fairorth, and Will Vauclain
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Featured Mathematician:
Bernhard Riemann (1826-1866)

Georg Friedrich Bernhard Riemann was a German mathematician who made huge strides in real analysis, differential geometry, and number theory. Although timid and afraid of speaking to large crowds at a young age, Riemann exhibited outstanding calculation abilities that his family and peers quickly noticed. He later met renowned mathematician Carl Friedrich Gauss at the University of Göttingen, who convinced him to drop theology and start studying in the mathematics department at the University of Berlin. Several years later, he began lecturing at the University of Göttingen and was promoted to head of the mathematics department.

Calculus students venerate Riemann for first concretely defining the integral, and engineering students routinely apply his discoveries in Fourier series and complex analysis to their work. In fact, Albert Einstein’s general theory of relativity would not have any basis without Riemannian geometry, a branch of differential geometry dealing with smooth manifolds in non-Euclidean space. His most enigmatic work was investigating the properties of the zeta function. By digging deep into this function, he formulated the ingenious Riemann hypothesis, which no mathematician has yet proven. The world celebrates Bernhard Riemann’s unbridled curiosity to uncover the mysteries behind the universe’s most perplexing questions, advancing both pure and applied mathematics for the greater good.
Pure Mathematics
Everyone has heard of “2D” and “3D,” though few ever give it much thought past what the words mean when associated with an image on a screen. In reality, nothing truly three-dimensional can be produced by a flat screen; this is only an illusion given by a two-dimensional image.

Understanding the idea of various spatial dimensions is simple. A one-dimensional surface would be a single line, along which a one-dimensional object could move anywhere but is limited to a single direction. Mathematics students are more familiar with the two-dimensional plane, with x and y axes. An object on a plane would have more freedom to move than on a line, but the possibilities are even more far-reaching in three-dimensional space. The most famous three-dimensional space is, of course, the universe. Objects can move along the x and y axes, but now they also move outward from the plane in the z direction. Although it is impossible to visualize the fourth spatial dimension, it is simply another addition to the number of directions that an object can explore: the x direction for the first dimension, y for the second, z for the third, and w for the fourth.

Understanding how an object in one set of dimensions looks in another can be tricky. Imagine a two-dimensional universe—a plane—being cut into an infinite number of one-dimensional lines. Now think of a two-dimensional object, perhaps a circle, moving along that plane. As it passes through a single one-dimensional slice, the circle appears as only a portion of the line, which grows until it hits the diameter of the circle, its widest length. The portion of the line then shrinks back down and disappears, exiting the slice. Now expand that puzzle to a three-dimensional space cut into an infinite number of two-dimensional planes. If a sphere rolls along the ground and passes through a plane, someone observing only that plane, slicing through the front of the sphere, first sees a circle floating off the ground. The sphere then grows until it just barely touches the ground, at which point the observed plane intersects the middle of the sphere. Finally, the circle shrinks back down before disappearing out of the observed plane.

Now imagine what a four-dimensional object looks like in a three-dimensional space, such as the universe. Since the universe is just one slice in four-dimensional space, a person can only see one slice of the four-dimensional object at a time. So the object’s orientation and actual shape changes depending on how a person sees it in three-dimensional space. It is not, for example, stuck with a specific number of sides. The object could appear to be changing shape, while in reality someone is just seeing different slices of the same unchanged object. The object could also float in air because its weight is supported by a neighboring three-dimensional slice.

Boundaries crumble when spatial dimensions increase. Think of a circle on a plane enclosed in a large rectangular box. The circle cannot leave the box because it only moves along
the x and y axes. However, if it moves along the z axis, moves along the x and y axes, and moves back along the z axis to its original position, the circle would be outside of the box. Similarly, a four-dimensional object can escape a locked box just by moving in the w direction.

By increasing spatial dimensions, entirely new objects are possible. The Klein bottle, a bottle with no edges and a single surface that never intersects itself, can only exist in four dimensions. In three dimensions, the neck of the bottle runs into the bottle itself and connects to the bottom of the bottle, linking the outside of the neck to the inside of the bottle and the inside of the neck to the outside of the bottle. Although the concept is interesting, the bottle has to intersect itself in three-dimensional space. However, by adding the fourth direction, the neck can move from the outside to the inside without ever contacting the surface.

The possibilities are endless in the fourth dimension. The universe may, in fact, just be one slice in an infinitely larger four-dimensional space. Though, if that were true, people would see more strange occurrences like floating and shape-shifting objects. Although traveling in the fourth dimension could not be achieved with today’s technology, doing so would shatter everything mankind holds true: the impossible would become possible.
Despite the various complaints high school students often make, humans are innately good at mathematics. Since prehistoric times, people have frequently made calculations and conjectures about the world around them. In fact, research has proven that toddlers are able to linearly count objects, compare sizes of sets, and subitize quantities before they can speak sentences of more than a few words. Those tasks sound complicated, but in reality, counting, comparing, and recognizing numbers are simple skills integrated into human nature.

Human understanding of mathematics began to accelerate around 3,000 BCE. Babylonians and Egyptians used basic arithmetic: they made tallies with unique numerals and advanced fractions to aid in trading goods and measuring land. From 600 to 300 BCE, the Ancient Greeks studied the applications of geometry and formalized systems to prove mathematical concepts. These discoveries all came from the desire to formulate abstractions from the natural world. An abstraction is a means by which specific concepts found in the real world can be generalized for broader applications. Basically, they are simple rules that allow people to understand more complex systems.

Even though people have always made abstractions, it took until 1889 for Giuseppe Peano, an Italian mathematician and historical linguist, to publish a list of the most basic ones. Concepts such as, “if a equals b, then b equals a,” seem obvious enough that they do not need to be explicitly defined. This innate understanding existed for a long time before people realized its importance. Simple concepts such as the one mentioned above are essential to mathematics, as they are the building blocks for every complex calculation. Peano is now commemorated for formally defining these basic rules of mathematics in the Peano Axioms. Peano saw a need to formalize these definitions when he and his contemporaries realized that all mathematical proofs could be derived from a series of simple statements. All of these statements are based on the understanding of natural numbers, positive whole numbers used for counting. Peano formalized eight of these statements in his axioms, dividing them into three categories: definition, equality, and sequence.
1. 0 is a natural number.

2. For every natural number x, x = x. Equality is reflexive.

3. For all natural numbers x and y, if x = y, then y = x. Equality is symmetric.

4. For all natural numbers x, y, and z, if x = y and y = z, then x = z. Equality is transitive.

5. For all a and b, if b is a natural number and a = b, then a is also a natural number. The natural numbers are closed under equality.

6. For every natural number n, the successor of n (denoted by S(n)), meaning the next number that follows n sequentially, or simply n + 1, is a natural number.

7. For all natural numbers m and n, m = n if and only if S(m) = S(n). S is an injection (a one-to-one function).

8. For every natural number n, S(n) = 0 does not exist. In other words, there is no natural number whose successor is 0.

Although it seems like these axioms have a lot of esoteric mathematical vocabulary, the concepts behind them are very simple. The first axiom outlines the definition of a natural number and includes zero in the set of natural numbers (since natural numbers can be used for counting, and it is possible to count zero things, zero can be allowed in the set).

The second, third, and fourth axioms describe relations in equality. A number is equal to itself; it is possible for two numbers to equal each other; and multiple equal groups of numbers are mutually equal. The fifth axiom connects the previous three to the idea of natural numbers, stating if a number is equal to a natural number, then it is itself a natural number.

The sixth, seventh, and eighth axioms expand the use of natural numbers from beyond recording amounts to counting with them. They state that natural numbers occur one after another; two natural numbers are equal if their successive natural numbers are also equal; and no negative numbers can be natural numbers (counting negative things is impossible).

When explained in meaningful, applicable terms, these axioms are easily understood. In fact, some of them seem almost blatantly obvious. After all, that is the point of axioms: to record the most basic concepts so they can be used more formally in complex mathematics.

The Peano Axioms have served as a powerful tool for mathematicians. It allows them to grasp the most basic facts behind a problem to understand the bigger picture, as the simplest ideas are often precursors to complex solutions.
The Riemann Zeta function is one of the most scrutinized complex analytical functions in history. Leonhard Euler discovered the function in 1737, and Bernhard Riemann was the first to test its properties. Riemann formulated a conjecture about the function’s zeros (known as the Riemann hypothesis), but nobody has yet proven it. In fact, proving the Riemann hypothesis is one of the seven Millennium Prize Problems chosen by the Clay Mathematics Institute. Riemann also studied how the Zeta function handles complex numbers as inputs. Looking at the function graphically lays a solid foundation for understanding how it deals with complex inputs.

The Riemann Zeta function is:

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

If \( s = 2 \), a real number, the function equals \( \pi^2/6 \):

\[ \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6} \approx 1.645 \]

There is a special reason for why \( \pi \) appears in this scenario, but it is beyond the scope of this piece. Because the function at \( s = 2 \) adds smaller components infinitely many times, regardless of what value is inputted, the function converges to a finite value. However, a small caveat appears once \( s < 1 \). The infinite series, rather than adding smaller numbers, adds increasingly larger numbers, which approach infinity. Thus, \( s < 1 \) are considered outside the domain of the function. For this article’s purpose, the graph of the function analytically continues past \( s < 1 \).

The function becomes interesting once complex numbers are inputted into the function. For real numbers, each input and corresponding output of the function can be thought of as a translation on the one-dimensional number line. But for complex numbers of the form \( x + iy \), each input and corresponding output can be thought of a translation on the two-dimensional complex plane. The \( y \) portion—the imaginary part—of a complex number places the complex number on the unit circle in the complex plane. The value of \( x \)—the real part—determines the rate at which the point rotates around the unit circle.
How does the Zeta function create an output from a complex input? Think of the real part of the complex numbers as piecing infinitely many lines together, whose lengths are the reciprocals of squares of numbers, and the imaginary part as rotating these lines. The function still converges, but only to a point on the complex plane. As long as $x > 1$, the complex output of the function is reasonable.

By understanding inputs and outputs of the Riemann Zeta function as transformations, the function can be graphed on the complex plane for all complex values $s = x + iy$ such that $x > 1$. However, the function seems incomplete without visually representing complex inputs $s$ with $x < 1$. So if the domain of the function is extended to the entire complex plane (a process called analytic continuation) such that all $s$ with $x > 1$ are mirrored to the set of complex numbers with $x < 1$, the beauty of the function begins to shine.

Figure 1: Analytic continuation of the Riemann Zeta function about $x = 1$
Series are one of the most broadly researched topics in mathematics. Most high school students learn them in their most basic form, but few realize their extensive potential. Calculators use Taylor series—infinite series of polynomials—to approximate a complicated answer such as $e^2$. Without this application of series, people would still be computing numbers by hand. Additionally, many real world phenomena can be represented by sequences and series in the same way that geometry can capture the essence of space. However, certain series that seem to violate human intuition have led mathematicians on a wild goose chase, leading to some shocking results.

The series under question is:

$$ S = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \ldots $$

$S$ sums all the natural numbers. To obtain an answer for this series, let’s first look at another series:

$$ S_1 = \sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \ldots $$

As more terms are added infinitely many times, the partial sums of $S_1$ oscillate between 1 and 0, so it seems that the series never converges. However, Italian mathematician Luigi Guido Grandi discovered a clever proof in 1703 to obtain an exact answer for $S_1$.

If,

$$ S_1 = 1 - 1 + 1 - 1 + \ldots $$

Then,

$$ 1 - S_1 = 1 - (1 - 1 + 1 - 1 + \ldots) $$
By distributing the negative sign into the parentheses, Grandi found:

\[ 1 - S_1 = 1 - 1 + 1 - 1 + \ldots \]

The right side of the equation is the same as the original series. Therefore,

\[ 1 - S_1 = S_1 \]

After combining the two \( S_1 \) terms, he ended up with the equation:

\[ 1 = 2S_1 \]

Which means that:

\[ S_1 = \frac{1}{2} \]

This is a shocking result, as none of the partial sums of \( S_1 \), which oscillate between two numbers, equal \( \frac{1}{2} \). Odd results like this open the door for more interesting mathematical findings.

The next series of interest is as follows:

\[ S_2 = \sum_{n=1}^{\infty} (-1)^{n+1}n = 1 - 2 + 3 - 4 + \ldots \]

Multiply \( S_2 \) by 2:

\[ 2S_2 = S_2 + S_2 \]

And obtain,

\[ 2S_2 = 1 - 2 + 3 - 4 + \ldots + 1 - 2 + 3 - 4 + \ldots \]

The statement above makes logical sense, as two times any quantity is equal to the sum of that quantity twice together. The second series is intentionally shifted to the right by one number (which is allowed by the commutative property of addition) to obtain the following ideal result:
\[
2S_2 = 1 - 1 + 1 - 1 + \ldots
\]

The value of the right side of the equation is already known, so,

\[
2S_2 = \frac{1}{2}
\]

Therefore,

\[
S_2 = \frac{1}{4}
\]

This is another surprising result. Just by looking at \(S_2\), it seems that it should grow increasingly large or small as numbers of larger magnitude are added.

Let’s now revisit the original series under question:

\[
S = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \ldots
\]

Subtract \(S_2\) from \(S\):

\[
S - S_2 = 1 + 2 + 3 + 4 + \ldots
- (1 - 2 + 3 - 4 + \ldots)
\]

Distribute the negative sign into the parentheses:

\[
S - S_2 = 1 + 2 + 3 + 4 + \ldots
- 1 + 2 - 3 + 4 - \ldots
\]

Add the two rows of numbers:

\[
S - S_2 = 4 + 8 + 12 + 16 + \ldots
\]
Half of the terms are cancelled out while the other terms are doubled. 4 can be factored out from the right side of the equation, yielding:

\[ S - S_2 = 4(1 + 2 + 3 + 4 + \ldots) \]

The series contained in the parentheses is equal to the original series, S, which can be substituted into the parentheses. Thus,

\[ S - S_2 = 4S \]

Combining like terms yields:

\[ -S_2 = 3S \]

Isolating S results in:

\[ S = \frac{-S_2}{3} \]

Plugging in the known value for \( S_2, \frac{1}{4} \), yields:

\[ S = \frac{-\left(\frac{1}{4}\right)}{3} \]

Therefore,

\[ S = -\frac{1}{12} \]

The sum of all the natural numbers is equal to -1/12. Many mathematicians are enthusiastic about this oddity, but others are opposed to it. The non-believers assert that this result violates two basic principles of mathematics:
1. Integers added together can never equal a fraction.
2. Positive numbers added together can never be negative.

This incredible series, which has captivated mathematicians for centuries, epitomizes how mathematics can spiral into controlled chaos, unveiling its beauty in a sea of disorder.
Aditya Sardesai '20

Gamma Function

Most high school students encounter an intriguing function in math class: the factorial. The textbook seems to ask the reader to yell the number three with excitement, but it is actually asking the student to multiply out $3 \times 2 \times 1 = 6$. The factorial is classically defined for integers $n \geq 0$ as:

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 2 \cdot 1$$

Although the definition of the factorial seems more boring than just yelling “three,” there is plenty to be excited about. It is useful in probability and sequencing problems, such as finding how many ways a group of students can sit in a classroom. But 18th century mathematician Leonhard Euler thought the definition was too limiting. The factorial function was defined only for integers $n \geq 0$, so he wondered if its definition could be extended to include a larger domain. Thus, the gamma function was born.

Euler allowed for the sequence $a_n$ to be:

$$a_n = \{1, 2, 6, 24, 120\ldots\}$$

Where $a_0 = 1$.

Euler found $a_n$ can be expressed in the form:

$$a_n = \frac{1 \cdot 2^n}{1 + n} \cdot \frac{2^{1-n} \cdot 3^n}{2 + n} \cdot \frac{3^{1-n} \cdot 4^n}{3 + n} \cdot \frac{4^{1-n} \cdot 5^n}{4 + n} \cdot \ldots$$

Using the absolute convergence for infinite products theorem and “Gauss’s criterion,” Euler found $a_n$ is equal to the following limit:

$$a_n = \lim_{k \to \infty} \frac{1}{1 + n} \cdot \frac{2}{2 + n} \cdot \frac{3}{3 + n} \cdot \ldots \cdot \frac{k}{k + n}$$

In Euler’s article “De progressionibus transcendentibus, seu quarum termini generales algebraeae dari nequeunt,” he shows $a_0 = a_1 = 1$ using the limit definition he derived. For $n = 2$, he writes:

$$a_2 = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \ldots$$
By algebraic rearrangement,

\[ a_2 = 2 \cdot \frac{2 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 3} \cdot \frac{4 \cdot 4}{4 \cdot 4} \cdot \frac{5 \cdot 5}{5 \cdot 5} \cdot \ldots \]

Therefore,

\[ a_2 = 2 \]

Similar, when \( n = 3 \), he writes:

\[ a_3 = \frac{2 \cdot 2 \cdot 2}{1 \cdot 1 \cdot 4} \cdot \frac{3 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 5} \cdot \frac{4 \cdot 4 \cdot 4}{3 \cdot 3 \cdot 6} \cdot \frac{5 \cdot 5 \cdot 5}{4 \cdot 4 \cdot 7} \cdot \ldots \]

By algebraic rearrangement,

\[ a_3 = 2 \cdot 3 \cdot \frac{3 \cdot 3}{3 \cdot 3} \cdot \frac{4 \cdot 4 \cdot 4}{4 \cdot 4 \cdot 4} \cdot \frac{5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5} \cdot \ldots \]

Hence,

\[ a_3 = 6 \]

By looking at his values of \( a_0 \), \( a_1 \), \( a_2 \), and \( a_3 \), \( a_n \) defined by his limit expression clearly mirrors the factorial function. But out of mere curiosity, Euler tried to compute \( a_{1/2} \) using his limit expression. Here is the result he obtained:

\[ a_{1/2} = \frac{1}{1 + \frac{1}{2}} \cdot \frac{2}{2 + \frac{1}{2}} \cdot \frac{3}{3 + \frac{1}{2}} \cdot \frac{4}{4 + \frac{1}{2}} \cdot \ldots \]

Thus, \( a_{1/2} \) can be rewritten as:

\[ a_{1/2} = \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9} \cdot \ldots \]

At first, it looks trivial to simplify the product above. But then Euler read a paper by John Wallis, who discovered:

\[ \frac{\pi}{4} = 2 \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{8 \cdot 10}{9 \cdot 9} \cdot \ldots \]

Thus,

\[ \frac{\pi}{4} = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2}{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2} \cdot \ldots \]
Using this identity, it is not difficult to conclude:

\[ a^{\frac{1}{2}} = \frac{\sqrt{\pi}}{2} \]

This is a stunning fact: factorials involve \( \pi \). In fact, a calculator displays an approximation of \( \sqrt{\pi}/2 \) when \( (\frac{1}{2})! \) is plugged in. Euler later found:

\[ a_n = \int_0^1 [\ln(x)]^n dx \]

As evident, this definite integral is well defined for positive and negative non-integer values of \( n \). Euler later discovered the integral above equals the improper integral below, which he named the gamma function.

\[ \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \]

The gamma function serves as the modern definition of the factorial function, characterized by a much wider domain than the classical definition, where:

\[ \Gamma(z) = (z - 1)! \]

At first glance, the gamma function just seems like an interesting quirk of mathematics, but it is useful in quantum physics, astrophysics, fluid dynamics, etc. For example, it can be used in statistics to calculate the time between occurrences of earthquakes. Regardless of its applicability, the gamma function is an emblem of the fascinating relationships that can be discover in mathematics through inquiry and exploration. It took only one person to wonder how to compute the factorial of a non-integer number; that curiosity gave life to a beautiful proof and elegant function.
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89… This is the famous Fibonacci sequence, created by Leonardo Pisano Bigollo, an Italian mathematician in the Middle Ages. Dubbed “Nature’s numbering system,” the Fibonacci sequence is embedded in the natural world, from the arrangements of a plant leaf to the bracts of a pinecone and the scales of a pineapple. Most people recognize the recursive formula of the Fibonacci sequence, \( a_n = a_{n-1} + a_{n-2} \), where \( a_1 = a_2 = 1 \).

But the sequence also has an explicit formula, one that designates the \( n \)th term of the sequence as:

\[
a_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}
\]

Or more simply,

\[
a_n = \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n
\]

This explicit formula is named Binet’s formula, in honor of French mathematician Jacques Philippe Marie Binet (the formula was actually known for more than a century before Binet was active).

To prove that this formula is true by induction, it must first be shown that the formula works for \( a_1 \) and \( a_2 \). Then it must be shown that the formula is true for \( a_k \) and \( a_{k-1} \), \( k \) being an arbitrary value greater than 2; thus, the formula would also work \( a_{k+1} \). Therefore, the formula would work for any positive integer \( n \).

Let’s first verify \( a_1 = 1 \) for this explicit formula.
Let's do the same process for $a_2$.

By foiling the numerator,

\[ a_1 = \frac{\left(\frac{1 + \sqrt{5}}{2}\right)^1 - \left(\frac{1 - \sqrt{5}}{2}\right)^1}{\sqrt{5}} \]

Thus,

\[ a_1 = \frac{\sqrt{5}}{\sqrt{5}} \]

Therefore,

\[ a_1 = 1 \]

Let's do the same process for $a_2$.

By foiling the numerator,

\[ a_2 = \frac{\left(\frac{1 + \sqrt{5}}{2}\right)^2 - \left(\frac{1 - \sqrt{5}}{2}\right)^2}{\sqrt{5}} \]

\[ a_2 = \frac{1 + 2\sqrt{5} + 5}{4\sqrt{5}} - \frac{1 - 2\sqrt{5} + 5}{4\sqrt{5}} \]

Therefore,

\[ a_2 = \frac{4\sqrt{5}}{4\sqrt{5}} \]
Hence, \[ a_2 = 1 \]

So this explicit formula satisfies the condition \( a_1 = a_2 = 1 \). Next, let's do the inductive proof, where \( a_{k+1} = a_k + a_{k-1} \) must be shown for any \( k \) using the explicit formula.

\[ a_{k+1} = a_k + a_{k-1} \]

Substitute in the explicit formula for \( a_k \) and \( a_{k-1} \):

\[
a_{k+1} = \frac{\left( \frac{1+\sqrt{5}}{2} \right)^k - \left( \frac{1-\sqrt{5}}{2} \right)^k}{\sqrt{5}} + \frac{\left( \frac{1+\sqrt{5}}{2} \right)^{k-1} - \left( \frac{1-\sqrt{5}}{2} \right)^{k-1}}{\sqrt{5}}
\]

Combine fractions with a common denominator:

\[
a_{k+1} = \frac{\left( \frac{1+\sqrt{5}}{2} \right)^k + \left( \frac{1+\sqrt{5}}{2} \right)^{k-1} - \left( \frac{1-\sqrt{5}}{2} \right)^k - \left( \frac{1-\sqrt{5}}{2} \right)^{k-1}}{\sqrt{5}}
\]

Factor:

\[
a_{k+1} = \frac{\left( \frac{1+\sqrt{5}}{2} + 1 \right)(1+\sqrt{5})^{k-1} - \left( \frac{1-\sqrt{5}}{2} + 1 \right)(1-\sqrt{5})^{k-1}}{\sqrt{5}}
\]

Combine numbers via a common denominator:

\[
a_{k+1} = \frac{\left( \frac{3+\sqrt{5}}{2} \right)(1+\sqrt{5})^{k-1} - \left( \frac{3-\sqrt{5}}{2} \right)(1-\sqrt{5})^{k-1}}{\sqrt{5}}
\]

Substitute the coefficients:

\[
a_{k+1} = \frac{\left( \frac{1+\sqrt{5}}{2} \right)2(1+\sqrt{5})^{k-1} - \left( \frac{1-\sqrt{5}}{2} \right)2(1-\sqrt{5})^{k-1}}{\sqrt{5}}
\]

Combine like exponential bases:

\[
a_{k+1} = \frac{\left( \frac{1+\sqrt{5}}{2} \right)^{k+1} - \left( \frac{1-\sqrt{5}}{2} \right)^{k+1}}{\sqrt{5}}
\]

This equation matches perfectly with the explicit formula for \( a_{k+1} \), proving that Binet's formula is true for all positive integers \( n \).

Q.E.D.
The Pythagorean theorem is fundamental to Euclidean geometry, stating that the sum of the squares of the two legs of a right triangle equals the square of the hypotenuse. In short, \( a^2 + b^2 = c^2 \).

A Pythagorean triple is a set of three positive integers \( a, b, \) and \( c \) such that \( a^2 + b^2 = c^2 \). The Pythagorean theorem implies that every triangle whose sides form a triple—a Pythagorean triangle—must be a right triangle. The converse of this statement does not necessarily hold (a triangle with \( a = 1, b = \sqrt{3} \) and \( c = 2 \) is a right triangle, but not all the sides have positive integer measures). By definition, a Pythagorean triangle has sides with measures that form a Pythagorean triple. When \( a, b, \) and \( c \) are coprime (\( a, b, \) and \( c \) are all prime), the triple is a primitive Pythagorean triple. For example, 5, 12 and 13 is a primitive triple, but 6, 8 and 10 is not because all three numbers share a common factor of 2. The latter is simply a scalar multiple of the most famous triple, 3, 4, and 5.

There are 16 different primitive Pythagorean triples where \( c < 100 \).

\[
\begin{align*}
(3, 4, 5) & \quad (5, 12, 13) & \quad (8, 15, 17) & \quad (7, 24, 25) & \quad (20, 21, 29) & \quad (12, 35, 37) \\
(9, 40, 41) & \quad (28, 45, 53) & \quad (11, 60, 61) & \quad (16, 63, 65) & \quad (33, 56, 65) & \quad (48, 55, 73) \\
(13, 84, 85) & \quad (36, 77, 85) & \quad (39, 80, 89) & \quad (65, 72, 97)
\end{align*}
\]

In an Algebra II class, students were tasked with generating examples of Pythagorean triples, and several came back with an interesting concept. First, let \( a \) be any positive odd integer. Then set:

\[
b = \frac{(a^2 - 1)}{2}
\]

\[
c = b + 1
\]

This guarantees \( a, b, \) and \( c \) satisfy \( a^2 + b^2 = c^2 \). Let’s verify this fact. Let \( a = 7 \), so \( b = 24 \) and \( c = 25 \). The set 7, 24, and 25 is, in fact, a Pythagorean triple. The students discovered this fact holds when \( a \) is any positive even integer.

Next, they set:

\[
b = \left( \frac{a}{2} \right)^2 - 1
\]

\[
c = b + 2
\]
Again, let's verify this fact. Let $a = 12$, so $b = 35$ and $c = 37$. This set is another Pythagorean triple. If $a = 6$, the scheme renders $b = 8$ and $c = 10$, producing the non-primitive triple mentioned earlier.

After these discoveries, the class explored the concept more deeply. The students shortly realized some trivial triples that satisfied the formulae they described cannot exist. For example, $a = 1$, 2, and 4 do not produce triangles. Their formulae also do not yield several known Pythagorean triples (i.e. 21, 20, and 29). The students did not want to produce Pythagorean triples that were simply multiples of primitive ones, so they searched for a method to find only primitive Pythagorean triples. At last, they found Euclid's formula!

According to Euclid, for any integers $m > n > 0$, $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ form a Pythagorean triple. Furthermore, Euclid guarantees $m$ and $n$ are coprime and not both odd if and only if the triple generated via his formula is primitive. If both $m$ and $n$ are odd, then $a$, $b$, and $c$ are all even and, thus, not coprime. To generate a primitive Pythagorean triple, when $m$ and $n$ are both odd, simply divide the generated $a$, $b$, and $c$ by 2. $m$ and $n$ cannot both be even, as they would violate Euclid's coprime condition; hence, $m$ or $n$ must be odd while the other even.

One result is immediately evident: the number of Pythagorean triples is infinite, as every combination of positive integers $m > n$ generates a Pythagorean triple. Every scalar combination of a Pythagorean triple is also a Pythagorean triple. Finally, there are infinitely many coprime integers $m$ and $n$, both of which are not odd. So the number of primitive Pythagorean triples is infinite.

Let's show that Euclid's formula for generating Pythagorean triples works for any arbitrary integers $m$ and $n$, where $m > n > 0$. Knowing $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, let's check if $a^2 + b^2 = c^2$.

$$a^2 + b^2 = c^2$$

Substitute:

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$$

Distribute:

$$m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4$$

Combine like terms:

$$m^4 + 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4$$

Both sides of the equation above are equal:

$$\therefore a^2 + b^2 = c^2$$

Thus, Euclid's formula works for any arbitrary $m$ and $n$ meeting the specified conditions.

Let's now show if there is a primitive triple of positive integers $a$, $b$, and $c$ such that $a^2 + b^2 = c^2$, then $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$. Since $a$, $b$, and $c$ form a primitive triple, each of $a$, $b$, and $c$ are pairwise coprime. Because $a$ and $b$ are coprime, at least one of them must be odd. By convention, suppose $a$ is odd; and if it is not, switch the positions of $a$ and $b$ so that it is. Having established the odd parity of $a$, $b$ must be even and $c$ must be odd because if both $a$ and $b$ were odd, then $c$ would have to be even. These conclusions can be made because the square of an odd integer is always odd, and adding two odds produces an even; furthermore, the square root of an even number is always even. All of these conclusions are discerned from the formula $a^2 + b^2 = c^2$. 

By algebraic rearrangement,

\[ a^2 + b^2 = c^2 \]

Factor:

\[ c^2 - a^2 = b^2 \]

Dividing yields the following, with \( m \) and \( n \) expressed in lowest terms:

\[
\frac{c + a}{b} = \frac{b}{c - a} = \frac{m}{n}
\]

It follows that:

\[
\frac{c + a}{b} = \frac{m}{n} \quad \frac{c - a}{b} = \frac{n}{m}
\]

By the distributive property,

\[
\frac{c}{b} + \frac{a}{b} = \frac{m}{n} \quad \frac{c}{b} - \frac{a}{b} = \frac{n}{m}
\]

By elimination,

\[
\frac{c}{b} = \frac{1}{2} \left( \frac{m}{n} + \frac{n}{m} \right) \quad \frac{a}{b} = \frac{1}{2} \left( \frac{m}{n} - \frac{n}{m} \right)
\]

Combine numbers via a common denominator:

\[
\frac{c}{b} = \frac{m^2 + n^2}{2mn} \quad \frac{a}{b} = \frac{m^2 - n^2}{2mn}
\]

Because \( m/n \) is in its lowest terms, \( m \) and \( n \) are coprime and at least one is odd. However, as demonstrated earlier, one must be odd and the other even. If both \( m \) and \( n \) were odd, then the numerator of \( a/b \) would be a multiple of 4 and the denominator would be a multiple of 2, implying \( a \) is an even integer—a violation of the condition that \( a \) is odd. Accordingly, the numerators of both \( a/b \) and \( c/b \) are odd and are reduced to their lowest terms. Therefore, \( a = m^2 - n^2 \), \( b = 2mn \), and \( c = m^2 + n^2 \) are all true.

Q.E.D.
When $n^k - n$ is Divisible by $k$

Grant Sterman ’18

$1^5 - 1 = 0, 2^5 - 2 = 30, 3^5 - 3 = 240, etc...$

Interestingly, it seems all the expressions above are divisible by 5. But a few examples do not prove the general case $n^5 - n$ is divisible by 5.

So let’s prove $n^5 - n$ is divisible by 5.

The proof by induction is rather simple:

Step 1: Let $F(n)$ be the statement $n^5 - n$ is divisible by 5. Suppose $F(k)$ is true for a positive integer $k$.

Step 2: Show for $k = 1$, $1^5 - k$ is divisible by 5. This is the initial and trivial situation, as $1^5 - 1 = 0$, and, of course, 5 divides 0.

Step 3: Show that $F(k)$ implies $F(k + 1)$. Thus, $F(k)$ is true for all positive integers.

$F(k + 1)$ says that $(k + 1)^5 - (k + 1)$ is divisible by 5.

\[
(k + 1)^5 - (k + 1) = (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k - 1)
\]

\[
= k^5 + 5k^4 + 10k^3 + 10k^2 + 4k
\]

\[
= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k - k
\]

\[
= (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k)
\]

\[
= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)
\]

\[
= 5m_1 + 5m_2
\]

$m_1$ and $m_2$ are simply any numbers; their values are unimportant as long as both are multiplied by 5. The left addend is divisible by 5 by $F(k)$ (induction in step 2), and the right addend is also clearly divisible by 5.
\[ 5m_1 + 5m_2 = 5(m_1 + m_2) = (k + 1)^5 - (k + 1) \]

\((k + 1)^5 - (k + 1)\) can be equated to the sum of two integers multiplied by 5. Therefore, \((k + 1)^5 - (k + 1)\) is divisible by 5. This means \(F(k)\) implies \(F(k+1)\). Thus, \(F(n)\) is true for all positive integers \(n\).

But there is also a more challenging proof without induction.

Consider the following expression:

\[
\begin{align*}
n^5 - n &= n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = n(n - 1)(n + 1)(n^2 + 1) \\
&= n(n - 1)(n + 1)(n^2 - 4 + 5) \\
&= n(n - 1)(n + 1)(n^2 - 4) + 5n(n - 1)(n + 1) \\
&= n(n - 1)(n + 1)(n - 2)(n + 2) + 5n(n - 1)(n + 1) \\
&= (n - 2)(n - 1)(n)(n + 1)(n + 2) + 5n(n - 1)(n + 1) \\
&= 5m_1 + 5m_2 \\
&= 5m_3
\end{align*}
\]

The left addend is divisible by 5 because the product of any 5 consecutive integers must be a multiple of 5. When multiplying five consecutive integers, it is inevitable at least one of those numbers is divisible by 5, so the whole product is divisible by 5. The right addend is a multiple of 5 explicitly. Therefore, the sum of the two addends is also divisible by 5, so \(n^5 - n\) is divisible by 5 for all positive integers \(n\).

It is true that \(n^1 - n\) is divisible by 1; \(n^2 - n\) is divisible by 2; and \(n^3 - n\) is divisible by 3 for all positive integers \(n\). Can this case be generalized to all positive integers \(k\), the exponent of the first term? If not, for which positive integers \(k\) is it true?

But there is an immediate roadblock: \(n^4 - n\) is not divisible by 4 for all positive integers \(n\). For example, \(3^4 - 3 = 78\), which is not a multiple of 4. Furthermore, for \(k = 6\) and \(n = 5\), \(5^6 - 5 = 15,620\) is not a multiple of 6. By looking at these examples, it can be conjectured that for even positive integers \(k\), \(n^k - n\) is not necessarily divisible by \(k\).

Odd integer values of \(k\) also paint an interesting picture. Because \(2^9 - 2 = 510\) and \(3^9 - 3 = 19,680\), neither of which is divisible by 9, it can be concluded that \(n^9 - n\) is not always divisible by 9. But when \(k = 7\) and \(n = 4\), \(4^7 - 4 = 16,380\), which is divisible by 7 \((7 * 2,340 = 16,380)\). With these discoveries, it can be conjectured that \(n^k - n\) is divisible by \(k\) for all positive prime \(k\). It turns out this is actually correct.

Consider \(n^p - n\) for some positive prime \(p\), which must be divisible by \(p\). It must be shown that the divisibility holds for all positive integers \(n\). This expression holds its divisibility when \(n = 1\), the trivial case. To prove divisibility for all positive integers \(n\), it must be shown that \((n + 1)^p - (n + 1)\) is also divisible by \(p\) whenever \(n^p - n\) is.
Let’s expand the following expression:

\[(n + 1)^p - (n + 1)\]

According to the binomial theorem,

\[(n + 1)^p - (n + 1) = n^p + \sum_{i=1}^{p-1} \binom{p}{i} * n^{p-i} + 1 - (n + 1)\]

\[= n^p - n + \sum_{i=1}^{p-1} \binom{p}{i} * n^{p-i}\]

\[= pm_1 + pm_2\]

The left addend is already known to be divisible by p; the right addend is a multiple of p explicitly. Therefore, the sum of the two is divisible by p.

\[pm_1 + pm_2 = pm_3\]

Thus, \(n^p - n\) is divisible by all positive prime p for all positive integers n.

Q.E.D.
Applied Mathematics
Look at any fantasy sports league with a significant prize pool. Anyone can figure out that the winners of that league use statistical analysis to pick their players. In fantasy sports, players earn points based on the statistics that their athletes have for each game they play in, so picking the athletes with the largest likelihood of acquiring good statistics each game is essential for winning.

Baseball is the gold standard for statistical analysis in sports; it the best sport for examining how statistics can boost players’ chances of winning their fantasy league. The MLB’s 162 games in the regular season provides a large sample size for athletic performance, increasing the chance of good athletes outperforming bad athletes and maximizing the amount of data available for each one. Besides those benefits, baseball is a game that provides abundant information about athlete performance, making it easy to analyze and determine the best-performing athletes.

Detailed statistics are available for every athlete and every play of the game. From there, a statistician’s job is to analyse the statistics and extract relevant pieces of information, a process known as sabermetrics. Amateur fantasy baseball players may look at the statistics on ESPN.com and choose athletes accordingly, but those statistics often misrepresent how a game actually went down. As Simon Singh notes in the sabermetrics chapter of *The Simpsons and their Mathematical Secrets*, an error is recorded for a fast athlete who runs for the ball but bobbles it rather than catching it, while no error is recorded for a slow athlete who lets the ball go over his head. If players were to pick their athlete solely off of who recorded the error, they would neglect the fact that the first athlete is actually the better choice because he was more likely to catch the ball. This example reveals just one of many reasons why statisticians continually invent new metrics to evaluate baseball players.

The popular metric OPS (On-base plus Slugging Percentage) determines the quality of a hitter. Traditionally, two of the statistics reported for hitters is slugging percentage and on-base percentage. A high slugging percentage for hitters means that they hit the ball hard and record bigger plays, while a high on-base percentage means that they get on base more often. An ideal fantasy hitter would do both of these well, so OPS is the sum of these two percentages, allowing statisticians to measure the overall effectiveness of a hitter. OPS provides a more complete picture of a hitter’s performance so that fantasy players select better athletes for their teams.

For pitchers, there is a similar dilemma. The traditional metric for a pitcher’s performance is ERA (Earned Run Average), which measures the average number of runs allowed for every nine innings pitched. The problem with ERA, statisticians have found, is that a pitcher’s ERA one year is a poor indication of their ERA the following year. This is because ERA is also dependent on external factors, such as the quality of the fielders in minimizing the effects of good hits. So, some statisticians turn to FIP (Fielding Independent Pitching) instead of ERA. This statistic is based solely off of actions that a pitcher can control: strikeouts, hit batters, walks, and home runs. Therefore, FIP paints a better picture of a pitcher’s performance. When solely analyzing ERA, a fantasy baseball manager might miss excellent pitchers that are hindered by bad fielders on their team, whereas FIP would allow the pitchers’ individual performance to shine.

This article analyzes just a few baseball statistics to demonstrate the effectiveness of applying sabermetrics to fantasy baseball. Browse the internet to learn more about how to apply statistics to fantasy sports. One helpful website is https://www.lookoutlanding.com/sabermetrics-101, which features articles introducing the reader to the basics of sabermetrics. Even though baseball seems to have the largest following of statisticians due to the large amount of data available from each game, other major sports have plenty of information available for fantasy applications, too.
Soft robotics is an emerging field of robotics that emphasizes the use of “soft” materials, such as polymers, elastomers, and silicone, rather than “hard” materials like metals and plastics. Materials in soft robots aim to replicate materials found in organisms or materials that can interact with the body. Soft robots move by pumping air or other fluids to contract or expand actuators, making them more flexible and adaptable to their surroundings. This means they can work in areas not accessible by rigid machines.

Traditionally, silicone materials have dominated the field of soft robotics. But now engineers need biodegradable and biocompatible actuators for soft robots to easily interact with the human body. The Soft Robotics Team at Haverford (consisting of Sixth Formers Xavi Segal, Matthew Baumholtz, Cal Buonocore, and Kyle Wagner, Fifth Formers Henry Sun and Intel Chen, Fourth Former Aditya Sardesai, and Second Former Bram Schork) created and developed biodegradable, edible gummy actuators. These actuators are an interactive candy for children and the precursor to implantable soft robotic devices.

Inspired by gummy worms, the students wondered if they could make gummy worms wiggle around like actual worms. This curiosity drove them to launch a project combining children’s candy and robotics design. The end goal was for the actuators to taste good and be able to complete at least three actuations.

As a proof of concept, the students tried to hollow out a cavity inside a gummy worm to see if it would inflate. When it successfully inflated, they realized gummy candy could be used to create actuators. They first melted down gummy bears and casted them in actuators molds. But two issues arose:

1. The gummy candy solution was extremely viscous, making it difficult to pour.
2. After curing, the candy solution adhered to the mold, breaking the mold upon removal.

To remedy these two issues, the students started using a 60 mL syringe to fill the molds and sprayed the molds with PAM cooking spray. Using a syringe allowed the gummy candy to flow out more smoothly and evenly; adding the PAM cooking spray allowed the candy to be molded without adhering to the mold, allowing the students to remove the solidified candy in one piece.

The students were ecstatic about the results they obtained with the gummy bear actuator, but they saw areas to improve on. The solidified gummy solution did not perform at the high level required for quick, easy production and rigorous actuation. It also had low elasticity, meaning it would not come back to its original shape quickly. When actuated, it would remain stretched out for long periods of time. Another problem was how much time the curing process to make each actuator took. Once the gummy solution was poured into the mold, the students had to wait for days before it even became semi-stale and elastic. This waiting period was elongated by having to attach the bottom layer of the solidified gummy solution to the top layer—a long wetting and melting process. The students also performed mechanical compression testing on their various gummy casts to compare them with actual gummy bears. They noted that it took about three days for recasted gummy actuators to resemble actual gummy bears. This is because heat denatures, or unfolds, the proteins inside the gummy bears.
Disappointed with the results of their first trial, the students decided the next move was to create their own gummy formula with customized specifications. They looked at common ingredients in candies with desired properties and designed an experiment to help them identify the ideal formulation. The key ingredients, they discovered, were gelatin, corn syrup, water, and flavoring. After qualitative analysis, they realized the sugar and flavoring had a negligible effect on the performance of the actuators, so they added them back to the formula after narrowing down the potential ratios of water, gelatin, and corn syrup. By again using qualitative analysis, they tried to find a ratio that would stiffen the gummy solutions enough for actuations, but not too much such that human teeth could not bite through them.

Figure 1: Stress vs. strain curves of two gummy candy solutions heated to two different temperatures

<table>
<thead>
<tr>
<th>Label</th>
<th>H_2O (mL)</th>
<th>Corn Syrup (mL)</th>
<th>Gelatin (g)</th>
<th>Qualitative Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
<td>7.2</td>
<td>Globby / Glue-like / (Untestable)</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>15</td>
<td>7.2</td>
<td>Very Elastic / Tear Resistant / Viscous</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>20</td>
<td>7.2</td>
<td>Elastic / Tear Resistant / Pourable</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>30</td>
<td>7.2</td>
<td>Elastic / Easily Torn / Low Viscosity</td>
</tr>
</tbody>
</table>

Figure 2: Table of gummy candy solutions with vastly different ingredient ratios

After qualitative analysis, they tried to find a ratio that would stiffen the gummy solutions enough for actuations, but not too much such that human teeth could not bite through them.

Figure 3: Stress vs. strain curves of gummy candy solutions with vastly different ingredient ratios

After testing general ratios (2, 3, and 4), the students tested specific ratios (3, 3.1, 3.2, 3.3, and 3.4) to hone in on their ideal specifications.

<table>
<thead>
<tr>
<th>Label</th>
<th>H_2O (mL)</th>
<th>Corn Syrup (mL)</th>
<th>Gelatin (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>30</td>
<td>10</td>
<td>7.2</td>
</tr>
<tr>
<td>3.2</td>
<td>25</td>
<td>15</td>
<td>7.2</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>20</td>
<td>7.2</td>
</tr>
<tr>
<td>3.3</td>
<td>15</td>
<td>25</td>
<td>7.2</td>
</tr>
<tr>
<td>3.4</td>
<td>10</td>
<td>30</td>
<td>7.2</td>
</tr>
</tbody>
</table>

Figure 4: Table of gummy candy solutions with close ingredient ratios

Figure 5: Stress vs. strain curves of gummy candy solutions with close ingredient ratios
The students chose Solution 3.2—dubbed “Fordmula”—for several reasons. First, it had the steepest stress vs. strain curve, meaning it would create the stiffest material able to withstand actuation pressures. This way, the actuator would quickly come back to its original shape and perform many actuations. The solution was also clear and visually appealing, in contrast to the murky solutions with higher corn syrup concentrations. Lastly, the solution was not so stiff that it would be impossible to chew. It had a great balance of easy tearing with high tensile strength and elasticity.

Silicone is traditionally used for making actuators; however, it is neither biodegradable nor biocompatible. But researchers at Harvard University and Boston Children’s Hospital recently created a thin silicone heart sleeve, which helps the heart beat. The students wanted to make a biodegradable heart sleeve, but they could not because Haverford does not yet have the resources to test such a device. Instead, they developed a proof of concept with their Fordmula actuator to show that it is biodegradable. They conducted a degradation study, split into two experiments, to demonstrate how their product would degrade inside the body over time. In both experiments, the degradation was studied in a 8.0 g/L sodium chloride solution, based off commonly-used Tyrode’s solution.

![Figure 6: Pictures of normal gummy bear actuator vs. Fordmula actuator in solution for 30 minutes](image)

![Figure 7: Volume vs. elapsed time trend of the normal gummy bear actuator and Fordmula actuator suspended in solution.](image)

![Figure 8: Mass vs. elapsed time trend of the normal gummy bear actuator and Fordmula actuator suspended in solution.](image)
The results of this degradation study confirmed that the Fordmula actuator is biodegradable. Additional ingredients can be added to the Fordmula or ingredient ratios in the Fordmula can be altered to make it degrade at a faster or slower rate in the body. Coupled with its natural biocompatibility, the Fordmula actuator’s biodegradability makes it a strong candidate for use as soft robot component in the body.

After testing their Fordmula solution, the students wanted to see how effective it was as an actuator. They were amazed with how effectively the actuators worked. However, they were still not child-approved, so they asked lower school teachers to let their kids play with and eat the edible Fordmula actuators. Unsurprisingly, the kids loved the actuators, validating the team’s success.

These soft robotics actuators have numerous real-world applications. They can spark interest in elementary school students to explore robotics and STEM in a fun, interactive way. Additionally, the silicone heart sleeve mentioned earlier can only stay inside the body temporarily, so two surgeries are required—one to insert the device and one to remove it. If gelatinous and biodegradable materials are used instead of silicone, the patient would only need surgery. These gummy bear devices can also deliver vitamins, medicine, and vaccines to young children. There is even a future for similar actuators in pediatric oral surgery as a pleasant way for reaching into a child’s mouth.

The students had a lot of fun working on this project. They have filed a provisional patent on the Fordmula, US Patent Application Number 62/517,299, entitled “EDIBLE PNEUMATIC SOFT ROBOTIC ACTUATORS,” filed on June 9, 2017 by The Haverford School. For more information on the Fordmula, visit https://softroboticstoolkit.com/edible-actuators.
A classic mathematics thought experiment goes like this: two criminals are arrested and put in solitary confinement without any means of communication with each other. The prosecutor, lacking evidence, decides to bargain with the two. Each prisoner has the option to either remain silent (cooperate) or betray the other (defect). The three outcomes in this scenario are the following:

1. If both defect, each serves 2 years in prison.
2. If one defects and the other cooperates, the former is set free and the latter will serve 3 years in prison.
3. If both cooperate, each serves 1 year in prison.

The prisoners quickly realize that if one cooperates, the other may defect, forcing the former to serve the maximum sentence; but if the former defects, then he may be set free or serve only a moderate sentence.

This thought experiment is commonly referred to as the Prisoner’s Dilemma. Originally the brain child of Merrill Flood and Melvin Dresher of the Research and Development Corporation (RAND), the game was formalized and named in 1992 by Albert W. Tucker.

The Prisoner’s Dilemma is a famous example of a game studied in game theory, “the study of multiperson decision problems,” according to Robert Gibbons in his book Game Theory for Applied Economists. Game theory can be used to analyze anything that involves decision making, from tic-tac-toe to international politics.

Two binary factors define all games in game theory: how players make moves, and whether or not players know other players’ moves.

1. If all players move simultaneously, then the game is static.
2. If all players move sequentially, then the game is dynamic.
3. If all players know other players’ moves, then the game has complete information.
4. If all players are uncertain of other players’ moves, then the game has incomplete information.

Thus, four types of games exist: static games of complete information, dynamic games of complete information, static games of incomplete information, and dynamic games of incomplete information. This article only touches on static games of incomplete information.

The Nash equilibrium is defined as an outcome of a game where no player has an incentive to deviate from his chosen strategy after considering an opponent’s choice. A game can have multiple Nash equilibriums, one, or none at all. What does the Nash equilibrium have to do with the Prisoner’s Dilemma? It seems that either choice each prisoner could make is a “right” move, so the outcome of the game becomes increasingly unpredictable. Finding the Nash equilibrium is one way to figure out the most stable outcome of this dilemma.

Here is a chart of the outcomes of each prisoner’s potential decisions:
With the matrix representing the four possible outcomes of the game based on each prisoner’s choice, the dilemma is now much clearer.

For Prisoner 1:
- If Prisoner 2 cooperates, defecting would be the optimal choice because 0 > -1.
- If Prisoner 2 defects, defecting would be the optimal choice because -2 > -3.

Likewise, for Prisoner 2:
- If Prisoner 1 cooperates, defecting would be the optimal choice because 0 > -1.
- If Prisoner 1 defects, defecting would be the optimal choice because -2 > -3.

Both Prisoner 1 and 2 would choose to defect no matter what the other prisoner’s decision is; therefore, both defecting (-2, -2) is the Nash equilibrium of this problem.

In any system, the “stable” property of a Nash equilibrium becomes the convention over time. An interesting situation to look at is when two cars encounter a hypothetical intersection without a traffic light or stop sign.

For Driver 1:
- When Driver 2 drives, stopping would be the optimal choice because he does not want to crash.
- When Driver 2 stops, going would be the optimal choice because he wants to reach his destination faster.

Likewise, for Driver 2:
- When Driver 1 drives, stopping would be the optimal choice because he does not want to crash.
- When Driver 1 stops, going would be the optimal choice because he wants to reach his destination faster.

After analyzing the possibilities, one driver going while the other stops is clearly the Nash equilibrium. The introduction of traffic lights to this intersection would enforce the Nash equilibrium by making each driver take turns going and stopping, which the drivers would naturally already do.
The modern car is no longer the mechanical machine people used to know: it is a computer. Pressing the brake does not physically stop the car. A message is sent to an electronic control unit, which activates the mechanical measures to stop the car. Placing mechanical control in the hands of computers has given cars the potential to protect those inside. Many features today, such as anti-lock braking, airbag control, and power steering, would not be possible without computer involvement. However, computerizing cars creates new vulnerabilities—the internal workings can be compromised through hacking. But luckily, car companies have taken steps to resolve this problem.

*How does the modern car communicate?*

Exploring the nature of a modern car is important to understand how it can be hacked. Each part of a car, such as the brake, steering wheel, and transmission, is linked to an electronic counterpart called an Electronic Control Unit (ECU). ECUs control the mechanics of the car, but they do not operate alone. They must communicate. Without communication, many of the “smart” safety, convenience, and economic features in cars would not be possible. The network through which ECUs communicate is called a Controlled Area Network Bus (CAN Bus).

A CAN Bus, unlike most networks, has no central computer or hub of data transfer. Instead, it is merely made up of data lines connecting each ECU to the other. Messages are broadcasted by a change in voltage in the data lines. The CAN Bus is a series circuit, meaning that only one message at a time can be transmitted. Figure 1 illustrates a basic CAN Bus (this diagram is extremely simplified, as modern cars have much closer to 70 ECUs than 6).

![Diagram of a basic CAN Bus](image)

1, 2, 3, 4, 5, and 6 are ECUs. 7 represents the OBD ports (discussed later). 8 refers to the actual CAN Bus, represented by turquoise and purple lines.

What does a CAN Bus message look like?

A CAN Bus message appears to be a string of meaningless 0s and 1s, but that is how car manufacturers want it to appear. Understanding the structure of a message or, even worse, the function of each string of numbers is exactly what a hacker needs to hijack a car. The 0s and 1s are actually a code for the message being sent through the CAN Bus. There are numerous parts to a CAN Bus message, all of which are detailed below.
Each bit is a digit that represents part of a command. Most CAN messages are written in hexadecimal (base 16), meaning that each bit can either be a number 0-9 or a letter A-F.

Let’s now dive into the different parts of a CAN Bus message (refer to Figure 2):

- **SOF**: This bit signals the start of a message.

- **CAN-ID**: Also known as the Arbitration ID, this set of bits identifies the priority of the message waiting to transmit. If two messages want to transmit simultaneously (which happens extremely often), the Message Identifier determines which message is transmitted first. The CAN-ID is important for hackers to pay attention to so that their messages take priority on the CAN-BUS. Messages with a lower Arbitration ID take priority over ones with a higher Arbitration ID.

- **RTR**: This bit allows ECUs to request information from other ECUs.

- **Control**: Also known as the Check Field, this set of bits lets the receiving ECU know how much data is being sent.

- **Data**: This set of bits contains the actual data being sent.

- **CRC**: Short for Cyclic Redundancy Check, this set of bits ensures the integrity of the data.

- **ACK**: Short for Acknowledge, this set of bits indicates if the message passed the Cyclic Redundancy Check.

- **EOF**: The set of bits signals the end of the CAN Bus message.

For hackers, only three parts of a CAN Bus message are relevant: the CAN-ID, the Control, and the Data.

**How do I hack?**

There is a way to tap into a car’s CAN Bus that requires almost no effort: infiltrating the OBD ports (referred to in Figure 1). The OBD ports are usually located under or within 2 feet of the steering wheel. Figure 3 shows where to find the OBD ports.
Though the OBD ports are extremely useful for rehearsing hacking skills and reverse-engineering CAN Bus codes (more on this later), they are hardly beneficial for black hat hackers who want to compromise cars. It would be ridiculous for a black hat hacker to drive alongside a car, plug his computer into the OBD port, and then start sending fraudulent messages into the car’s CAN Bus.

Instead, hackers employ wireless methods to infiltrate a car’s CAN Bus. The first is through cellular networks. One of the most publicized car hacks took place when Charlie Miller and Chris Valasek (both white hat hackers) found a way to use the Sprint cellular network to connect to the telematics ECU of a car, which controls all the entertainment in the car (i.e. the radio) and collects information from cellular networking. Hacking a car’s entertainment system does not seem frightening, but because every ECU is connected to the other in a CAN Bus, Miller and Valasek were able to send fraudulent messages to more important ECUs, such as those in the engine and the brakes. Furthermore, because this hack was done over the Sprint cellular network, it could be performed remotely on any susceptible car anywhere in the country. Miller and Valasek ran a scan of all cars in the U.S. and found 470,000 cars that could be potentially compromised by their hack. Thankfully, after they performed the hack, car companies took steps to secure their vehicles, and Congress passed legislation forcing car manufacturers to meet stricter safety standards.

Another way of connecting remotely to a CAN Bus is through the car’s Wi-Fi. To access any Wi-Fi network, a hacker connects via a port. Ports make up a Wi-Fi network’s connections with the internet and its users. All incoming network traffic is assigned a port number to tell the router what kind of traffic is being sent. Ideally, a Wi-Fi router leaves no open ports at any time and opens up new ports only when it authorizes a new user to connect to the internet. However, when routers leave ports open, they open a door through which hackers can launch attacks and gain access to CAN Bus messages. Scanning for an open port is simple and can be done on a LINUX command prompt with the command “nmap.” As for running the actual Wi-Fi attacks, several free, open-source hacking tools such as Armitage and Burpsuite run the attacks and offer detailed explanations to the hacker of each attack.

I’m in, now what?

The hacker has finally bypassed the security system of the car and now has access to the car’s CAN Bus traffic. The only thing standing between him and full control of the car is reverse-engineering what each message says.
First, let’s look at what a CAN traffic interface may look like. Some hackers use a program called Vehicle Spy to view CAN traffic. Though it costs close to $5,000, it is extremely user-friendly. Shown in Figure 4 is a Vehicle Spy CAN interface in action.

![Vehicle Spy CAN Interface](image)

**Figure 4: Vehicle Spy CAN Interface**

Vehicle Spy parses each message and pulls out the useful bits. For example, it pulls out the CAN-ID (or Arbitration ID), which is important for writing fraudulent messages that take priority on the CAN Bus. The actual data from each message is displayed in the column DataBytes as a string of numbers.

Now the hacker must decode the data. The algorithms to translate message data into understandable commands are generally proprietary. Different hackers circumvent this obstacle in unique ways. At car hacking conventions, participants can create a program that receives real-time CAN Bus data and parses it to find patterns. Using previous knowledge and trial-and-error, some CAN Bus commands, such as the ones that start the car, can be reverse-engineered.

The final step of hacking is to enter fraudulent commands into the CAN Bus. Vehicle Spy and other hacking software allow a hacker to do this directly by simply inputting a CAN-ID and some data. This is by far the easiest step, and the hacker can finally see his work pay off.

Luckily, beginner hackers do not have to pay $5,000 for decent software before they can attempt to hack. There are several open source CAN Bus traffic interfaces, the most popular of which being Socket CAN, which conveniently displays on LINUX Command Prompt.

The world of hacking and computer security is large and constantly growing. This is only a taste of car hacking. There are so many other variations of attacks a hacker can run. Ordinary people are encouraged to try hacking: in the digital age, it always helps to know what you are up against.
Crime rates in the United States are approaching an all-time low, thanks to the work of seasoned criminologists. They are renowned for their in-depth analysis of visual evidence, but many people are unaware of the math and science behind their work. Forensic criminologists study the behavioral patterns of criminals by mathematically analyzing the time, geographic profiling, blood patterns, and several other pieces of evidence left behind at a crime scene. With this information, criminologists and police work together to prevent future crimes and even stop crimes already in progress.

The time of death is key information for a murder case. Not only does it help with collecting more data for behavioral profiling, but it can also help confirm witnesses’ claims and support or relieve suspects. Upon examination of the deceased, the medical analyst records both the temperature of the body and the surrounding environment (the air temperature). This data is plugged into the equation for Newton’s Law of Cooling, which states the rate at which heat is released from the body is directly proportional to the difference in temperatures between the body and its surroundings:

$$T(t) = T_e + (T_0 - T_e)e^{-kt}$$

- $T(t)$: Temperature of the body at a certain time
- $t$: Time after death, where $t = 0$ is when the deceased was killed
- $T_e$: Temperature of the surrounding environment (a constant)
- $T_0$: Starting temperature of the body (a constant at 37°C, the average human body temperature)
- $k$: Cooling constant that varies depending on the deceased’s body type

Newton’s Law of Cooling can be used to find the temperature of a human body at a specific time. Upon observation, the medical analyst records the body temperature in several places, including the forehead, behind the ear, under the arm, and the genitals. It is crucial that these temperatures are recorded as soon after death as possible to prevent the body from reaching the same temperature as its surroundings, at which time Newton’s Law of Cooling cannot extrapolate when the murder happened.

There are several factors that determine the value of $k$. First, the analyst measures and records the body’s fat percentage. Depending on the percentage, the $k$ value increases or decreases. If the fat percentage is high, the body retains more heat than if it were low, so the $k$ value should be greater. Another factor taken into consideration is the clothing covering the body. Thicker clothes such as sweatpants retain more heat, resulting in a higher $k$ value, and vice versa. After gathering all the constants, the criminologist isolates the unknown value $t$, the amount of time the body has been deceased.

A criminologist also examines blood stains at the crime scene. By analyzing blood patterns, questions about the crime can be answered: how many criminals were there, where the murder took place, what angle the bullet or knife came from, and whether or not the crime scene was altered in any way. To get an accurate read of the patterns, the medical examiner needs to be proficient in three disciplines: biology, physics, and—of course—mathematics.
In Figure 1, Exhibit A shows a blood droplet that hits the ground at 90°, giving it a uniform diameter, expressed by the variable d. In Exhibit B, the droplet comes into contact with the ground at an unknown angle less than 90°. Because the spherical droplet stretches upon impact with the ground, it coagulates into an ellipse. The criminologist measures the length (L) and the width (W) of the ellipse to calculate the angle from which the blood spilled by isolating \( \theta \) in the equation below:

\[
\sin(\theta) = \frac{W}{L}
\]

Once \( \theta \) is solved for, the criminologist uses the angles at which the blood droplets hit the ground to extrapolate where all the blood came from (refer to Exhibit C in Figure 1). This convergence point is then used to measure the distance D between the point and the blood stain to find the height from which the blood fell using the equation:

\[
H = D \tan(\theta)
\]

Where H is the height of the body that the blood spilled from.

Police use data from blood patterns to create a more accurate behavioral profiling chart. Crimes are usually committed repeatedly—either because of a perpetrator’s physiological or mental illness—so the police can narrow down the type of people prone to committing a crime and the neighborhoods that they live in. With this information, police can patrol high-risk areas, preventing crimes in the future.

Geographic profiling produces the most crucial information in a case. By stringing together several locations connected to crimes, police can detect geographic regions that have the highest probability of a crime being committed. The main question asked when determining geographic profiling is “How far is a criminal
willing to travel to commit a crime?” This is calculated through the equation:

\[
P(z, \alpha | x) = \frac{P(x | z, \alpha) \pi(z, \alpha)}{P(x)}
\]

- \(z\) = Location of the criminal’s home
- \(\alpha\) = Average distance a criminal is willing to travel
- \(x\) = Previous location where a criminal has committed a crime
- \(P(z, \alpha | x)\) = Region in which a criminal may travel from his home \((z)\) at an average distance \((\alpha)\) if he or she had previously committed a crime at \(x\)

In order to take into account several crimes, not just one, mathematicians have derived the following equation:

\[
P(z, \alpha | x_1, ..., x_n) = \frac{P(x_1, ..., x_n | z, \alpha) \pi(z, \alpha)}{P(x_1, ..., x_n)}
\]

Where \(P(x_1, ..., x_n)\) = Crimes committed in multiple locations

Finding and calculating these variables is known as Bayesian analysis. By analyzing the distribution of crimes and averaging the distances between them, police and investigators can calculate which neighborhoods are more susceptible to crime and place more officers on patrol there. With these mathematical formulae to analyze time of death, blood patterns, and geographic profiling, crime rates in the United States and around the world are decreasing to record lows, illustrating how the power of mathematics can improve everyone’s lives.

**Chaos Theory**

Nico Tellez ’18

If a butterfly flaps its wing in Brazil, there will be a tornado in Kansas. This is the romanticized Hollywood version of chaos theory, but it illustrates the concept that small changes can have profound effects. Consider a pinball machine. The ball is governed by gravity, elastic collisions, and the angle at which the paddles hit it, but the outcome of the game is unpredictable. Think about the stock market. There are thousands of people whose jobs are to analyze it. Top brokerage firms have created market simulations based on hundreds of years of advanced research. So why do these models always fail during a market crash? It is because the stock market is governed by chaos theory, making it impossible to predict.

Chaos theory was first discovered in 1961 by Edward Lorenz, a meteorology professor at the Massachusetts Institute of Technology. Over a three-month period, Lorenz created a computer model to simulate the weather. The model was based on 12 variables, including temperature and wind speed. He ran the simulation one time and was inclined to run it again. Before the second run, he rounded one of the variables from 0.506127 to 0.506. This small change had an alarming effect on the entire pattern his program produced. He found that small changes in the initial conditions of a dynamic system can have major effects on the system as a whole. After further tinkering, Lorenz realized that the results of his tiny alterations produced a pattern, so he could
Chaos Theory

not call his discovery random theory. Randomness implies no correlation, but he found a correlation in the outcomes after making small changes to the initial conditions. The pattern, resembling a butterfly, is illustrated below.

Although it is impossible to predict when a drastic change occurs, there is a clear pattern to the chaos. When tweaked, dynamic systems can morph into something completely different.

From pinball to the stock market, the world is unpredictable in countless ways. But chaos theory should instill a sense of hope in everyone: all it takes to change the world is one person or idea.

The Origin of Numbers

Toby Ma ‘20

Numbers are ubiquitous in modern society. How could anyone figure out who won last week’s basketball game without numbers? How could someone tell the time or date without numbers? And, most importantly, how could students and mathematicians worldwide solve algebraic equations without numbers? Today, nearly every country in the world uses the characters 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 to indicate quantities, breaking barriers of language, culture, and ethnicity. How did these ten characters dominate the world, and who invented them?

Before the dawn of civilization, humans had no need for numbers. Anything in the natural world could be counted using ten fingers, or, if necessary, tallied by drawing lines or notches in wood or stone. This system was easy when humans had very few items, like three dogs or five spears. However, once agriculture and cities developed across the planet, finger-counting and tallying did not suffice. The ruler of a city needed to remember that he had 253 cows and 764 pigs, numbers far exceeding the fingers on his hands, and drawing 1,017 tallies would have been too tedious. Humans had to devise another way to count.

Sumer, one of the first civilizations, was the first to develop a numeral system. The rulers of Sumerian cities needed to record many things, such as the amount of food harvested per year, foreign goods from rich trade networks, plundered treasures from conquests, and taxes. So they developed a method of assigning clay tokens to represent material goods and using markings to indicate various quantities. Eventually, accountants in these cities realized they could conveniently record markings on clay tablets, without using tokens. Thus, the first numeral system was born.
The ancient Egyptians were also busy developing their own numeral system. The Egyptians loved building monumental structures like pyramids and temples, and to build such large structures, they needed more precise instructions than “really, really tall” and “really, really big.” To accurately measure distances, the Egyptians assigned real-world lengths to these numbers, creating some of the world’s first units of measurement. They used the cubit, the length from someone’s elbow to the tip of his middle finger. Because the length of one’s forearm and hand varied from person to person, priests kept standardized cubit lengths.

In ancient China, people during the Shang dynasty carved their numeral system into tortoise shells and bones. The Chinese number system was multiplicative, so instead of a unique symbol for the number “200,” the symbol was a combination of the symbol for “2” and “100.”

Hundreds of years later, the Roman Empire began to expand around the Mediterranean. They developed a numeral system known today as—no surprise—Roman numerals. Instead of assigning symbols to numbers from one to nine, the Romans only created symbols for 1, 5, 10, 50, 100, 500, and 1,000 (I, V, X, L, C, D, M, respectively). They combined the symbols to add them up to a numerical value. Europe continued to use Roman numerals for several centuries after the fall of the Roman Empire. They eventually fell out of use with the introduction of Arabic numerals, but they still persist today for counting Super Bowls and numbering pages in the preface of textbooks.

Meanwhile, a civilization in Central America had developed a numeral system hundreds of years more advanced than the Roman and Chinese systems. The Mayans developed positional notation, where the placement of a digit within a number gives that digit an intrinsic value. For example, the digit “3” in 35 means “30,” while the “5” just means “5.” Other numeral systems at the time did not have positional notation, instead relying on adding values to each other. The Mayans also implemented a base-20 system, meaning that place values were based on powers of 20, instead of the base-10 system used today. A dot represented 1 and a long dash represented 5. They also created a symbol for 0, although it was mostly used as a placeholder instead of a value. The Mayans had an advanced number system, but it eventually died when the Mayan civilization collapsed in the 9th century CE.

Despite the misleading name, Arabic numerals, the numbers used worldwide today, first originated in India, where they first came into use in 500 BCE. The Indians had also developed positional notation, which was followed closely by the “invention” of 0, a crucial placeholder in positional notation. For the first time, humans expressed 0 as a value that meant nothing. This idea was linked to Hinduism, as scholars attempted to describe abstract aspects of the religion in mathematical terms.

Eventually, the Indian numeral system spread to the Islamic world to the west and was used by Arabic scholars to achieve many breakthroughs in math and science. The Hindu-Arabic numerals later spread to Islamic territories in Northern Africa. Fibonacci, the same mathematician whom the Fibonacci sequence is named after, traveled to North Africa and introduced the Hindu-Arabic numerals to Europe. Europeans then spread the numeral system across the world during the colonial period, making Hindu-Arabic numerals the most widely used numbers today.
One of the major goals of any company is maximizing output, which leads to higher potential profits. But how does a company do this? It would need to consider which factors contribute to greater output. Over years, economists have come up with various models on maximizing output, the most widely used being the Cobb-Douglas Production function.

In 1928, Charles W. Cobb and Paul H. Douglas released a study in which they modeled the growth of the American economy. They created a model that determined the economy’s output quantity based on two main factors: the amount of labor and the amount of capital investment. In other words, they created a simplified model of the output of the economy as a function of the amount of capital invested and labor involved. Labor refers to work that humans do, and capital investment refers to money invested in buildings, machinery, etc. to further businesses. This function, although intended to model the entire American economy, is now also used on a microeconomic scale to predict the output of individual businesses.

The Cobb-Douglas Production function is as follows:

\[ P(L, K) = bL^\alpha K^\beta \]

Where,
- \( P \) = Total production (the monetary value of all goods produced in a year)
- \( L \) = Labor (the total number of hours worked in a year)
- \( K \) = Capital input (the value of all machinery, equipment, and buildings)
- \( b \) = Total factor productivity (the portion of output not explained by the amount of inputs used in production). As technology increases, humans can get more output from a given input, which is taken into account by increasing the value of \( b \).
- \( \alpha \) and \( \beta \) = output elasticities of labor and capital, respectively. These values are constants determined by available technology.

How did Cobb and Douglas arrive at this function? First, they made a few assumptions about the relationship between \( L \) and \( K \):

1. If either labor or capital vanishes, then so does production.
2. The marginal productivity of labor is proportional to the amount of production per unit of labor.
3. The marginal productivity of capital is proportional to the amount of production per unit of capital.

By definition, the marginal productivity of labor is the rate at which production changes with respect to labor, and the marginal productivity of capital is the rate at which production changes with respect to capital. So let’s express these assumptions in an equation.

If \( P = P(L, K) \), then the partial derivative of \( P \) with respect to \( L \) is equal to marginal productivity of labor and the partial derivative of \( P \) with respect to \( K \) is equal to marginal productivity of capital. Using the second assumption Cobb and Douglas made, the following equation can be written:
Because only the rate at which L affects P is significant, K must remain constant (hence set $K = K_0$). Therefore, the previous equation can be modified to:

\[
\frac{\partial P}{\partial L} = \alpha \frac{P}{L}
\]

Solving this differential equation results in another equation. First, both sides of the differential equation are integrated:

\[
\int \frac{1}{P} dP = \alpha \int \frac{1}{L} dL
\]

Which yields:

\[
\ln(P) = \alpha \ln(cL)
\]

Therefore, by algebraic rearrangement:

\[
P(L, K_0) = C_1(K_0) L^\alpha
\]

Here, $C_1$ was set equal to constant $K_0$ because it varies based upon the value of $K_0$. What was just done can also be applied to the third assumption Cobb and Douglas made:

\[
\frac{\partial P}{\partial K} = \beta \frac{P}{K}
\]

Repeating the same process results in the following relationship:
Combining the two equations Cobb and Douglas found produces the following equation:

\[ P(L, K) = b L^\alpha K^\beta \]

Where \( b \) is a constant independent of both \( L \) and \( K \). The first assumption requires that \( \alpha > 0 \) and \( \beta > 0 \). If labor and capital are both increased by the same factor \( m \), then the following is true:

\[ P(mL, mK) = b (mL)^\alpha (mK)^\beta \]

Therefore,

\[ P(mL, mK) = m^{\alpha+\beta} b L^\alpha K^\beta \]

So the equation below is true:

\[ P(mL, mK) = m^{\alpha+\beta} P(L, K) \]

If \( \alpha + \beta = 1 \), then \( P(mL, mK) = mP(L, K) \), which implies that production is also increased by a factor of \( m \). Cobb and Douglas assumed that \( \alpha + \beta = 1 \), and hence they arrived at their famous function:

\[ P(L, K) = b L^\alpha K^{1-\alpha} \]
An exercise in ruthless power hungry complacency—gerrymandering. The nation is plagued with this practice in which each state’s majority party chops districts to ensure seats are non-competitive and favorable. Voters tend to congregate: liberals urban, conservatives rural. By grouping all opposition voters into one district, their political power is unfairly contained and neutralized. Then, any remaining opposition voters are siphoned out into districts with more favorable voters, ensuring a large quantity of safe, friendly districts. The result destroys democracy. First, politicians no longer fear reprimand from constituents because their seats become non-competitive. Second, a holistically competitive state can be converted overwhelmingly to one party without changing a single vote. Look at Pennsylvania. By total voters, the state is highly competitive, with only a +1% Republican lean. But after gerrymandering, Pennsylvania is solidly Republican at +44% (Republicans hold 13/18 congressional seats). This also happens in blue states like Massachusetts, which has been converted entirely to Democratic at +100% from only +40%.

Figure 1 is a simple diagram of how gerrymandering works:

![Figure 1: Theory behind gerrymandering](image)

Luckily, there is a solution to this crisis: basic arithmetic. Election efficiency can be tested by counting wasted votes. All votes that the winner receives beyond 50% would be counted as wasted votes, and all the losers votes would count as wasted because they failed to achieve their goal. Statewide, each party’s wasted votes would be tallied. The larger the gap between parties, the greater the gerrymandering. Courts currently do not have a system to prove or quantify gerrymandering, so these wasted-vote calculations could be adopted as a legal method to locate and strike down gerrymandered districts.
Once gerrymandering is identified, problematic districts can be redrawn to prioritize compactness, the current Supreme Court’s definition of district fairness. Districts would be drawn with the smallest length of lines possible while maintaining equal populations in each. This would remove subjective partisanship, lowering gaps in election efficiency.

Figure 2: Pennsylvania’s district lines now vs. ideal district lines
Fun Math
Across

The Mediterranean coast

He was seen.

Son of a wealthy merchant,

He learned of mathematics across the seven seas.

Bringing the Hindu-Arabic numeral system to Europe, he changed the Western world forever.

By studying generations of rabbits, he was able to discover what Indian mathematicians had known for centuries: the immortal Fibonacci sequence.

His discoveries transcended his own existence, transforming how we see mathematics, nature, and genetics. Derived from Fibonacci’s famous sequence came a golden ratio, influencing great marvels of architecture, painting, and design throughout the world.

Fibonacci

Winslow Wanglee '19
Young Ada was born in 1815
To a Lord and a Lady of no small esteem.
Her father, a poet, left when she was small,
Through war and disease, he met his downfall.

Her mother, it’s said, was a mathematician,
Who was, in some ways, a new inquisition
That steered Ada clear of her late father’s path,
And instead directed her focus towards math.

Inspiration came from a loom by Jacquard,
Which worked by punching some holes in a card,
And then the machine wove patterns from there
That were better than those made with practice and care.

At eighteen she heard of a weekly event
To which many thinkers of Ada’s day went.
Hosted by Babbage, inventor well known,
This party was first where his engine was shown.

A difference engine that could calculate
figures in series that it could create.
She was amazed by this brilliant machine
(Keep in mind, at this time, it was powered by steam).

Ada and Babbage, they made a great team,
When a few years later he told her his dream
Of making another invention far better
That performed operations right down to the letter.

He had a French paper on this new kind of engine
That Ada translated and, as Babbage mentioned,
She made many notes for the readers to see,
And her most insightful was labeled “Note G.”

Modern-day coding is based on this note,
Because of the logic and loops that she wrote.
She saw the potential to make greater things,
Using formulas to instruct small, toothèd rings.

Bernoulli’s large numbers, then, Ada realized,
Could be formulated and clearly derived
By using this engine to sum all the squares,
Or cubes or high powers; whatever one dares!

She then postulated this new kind of tech
Could make music and graphics with all kinds of specs.
Mathematical wizardry, her big breakthrough
Would help Alan Turing to win World War Two.
Ken Keeler and the Futurama Wheel

Will Vauclain ‘19

Got his degree from Harvard,
    A PhD in Math.
A writer for the Simpsons,
    And Futurama, too.

Then one day came a story,
    It needed math, so true,
A problem needed solving,
    And Ken took up the task.

Imagine switching bodies,
    But only once per pair;
Imagine all the trouble,
    When done while lacking care.

How can you ever get back
    To your beginning form?
Ken did the math on this one,
    He made a theorem too:

It says, just summon two more,
    And that there is the cure;
Just use their brains like buffers
    To sort back all the minds.
The Billiard Ball Problem

Billiard balls bouncing at different angles, 
On different paths, different tables.
Endless combinations. Where do the balls go? 
The problem, seemingly simple, unwinds, 
Revealing math's deepest secrets.

The solution eluded great mathematicians. 
But Mirzakhani, a woman in Iran, 
Denied the most fundamental of rights, 
Transcended the best, solving the problem, 
Developing the modern mathematician's toolbox.

Painting math upon canvases like an artist, 
Developing moduli space theory, 
The key to quantum field theory, 
Riemann surfaces, 
The base of general relativity.

All fields feel her impact. 
Now the first woman and Iranian 
To win a Field Medal, 
Let her rest in peace.

“I think it's rarely about what you actually learn in class . . . it's mostly about things that you stay motivated to go and continue to do on your own.”
~Maryam Mirzakhani

Maryam Mirzakhani (1977-2017) was the first woman to receive the Fields Medal, the most prestigious award in mathematics. Sadly, she lost her battle to breast cancer in July 2017.
If you are interested in contributing to Newton’s Notebook in 2018-19, contact Mickey Fairorth, Will Vauclain or Charlie Baker, the Editors-in-Chief for Volume III.